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Hilbert and Hilbert–Samuel Polynomials and Partial Differential Equations

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Abstract—Systems of linear partial differential equations with constant coefficients are considered. The spaces of formal and analytic solutions of such systems are described by algebraic methods. The Hilbert and Hilbert–Samuel polynomials for systems of partial differential equations are defined.

Key words: *system of linear partial differential equations, space of formal solutions, space of analytic solutions, symbol of a system, Hilbert polynomial, Hilbert–Samuel polynomial.*

1. INTRODUCTION

We consider general systems of linear homogeneous partial differential equations with constant coefficients for one unknown function z of the complex variables $x_1, \dots, x_n$. The *symbol* of such a system is the algebraic variety M in the dual space with variables $\xi_1, \dots, \xi_n$ whose ideal $\mathcal{I}$ is generated by the polynomials obtained from the equations of the system by replacing differentiation with respect to the variables x_i with multiplication by the corresponding variables ξ_i. We study the relation between the algebraic variety M and the spaces of formal and analytic solutions of the initial system of differential equations.

One of the basic invariants of an algebraic variety is its Hilbert function (see, e.g., [1]). This is a function H of positive integer argument; to each k it assigns the dimension of the quotient space of polynomials of degree at most k by the vector subspace consisting of the polynomials belonging to the ideal $\mathcal{I}$. Hilbert's celebrated *Nullstellensatz* says that the function H is polynomial for sufficiently large positive integers. The degree of this polynomial is equal to the dimension r of the variety M, and the highest coefficient multiplied by $r!$ is the degree of the variety M (i.e., the number of intersection points of M with a general affine plane of complementary dimension, counting multiplicities). How is the Hilbert polynomial symbol of M related to the initial system of differential equations? This paper answers this question.

Given a point u and a positive integer k, consider the vector spaces $\mathcal{O}_u(k)$ and $F_u(k)$ of the k-jets of analytic and formal solutions of the system at u. We prove that the dimensions of these spaces coincide, do not depend on the point u, and are equal to the value $H(k)$ of the Hilbert function of the system symbol at k (see Theorem 2 and Proposition 2). The proof proceeds as follows. First, we give an algebraic description of the space of formal solutions of the system at the point u (Corollary 2). This description immediately implies $\dim F_u(k) = H(k)$. Then, we prove the following approximation theorem (Theorem 1): *For any formal solution of the given system at a point u and any positive integer k, there exists a quasipolynomial solution of the system* (*i.e., a linear combination of products of polynomials and exponentials of linear functions*) *which has the same k-jet as the given formal solution.* This readily implies the equality $\dim F_u(k) = \dim \mathcal{O}_u(k)$.

The Hilbert–Samuel function is a local invariant of an algebraic variety (see, e.g., [1]). Hilbert's *Nullstellensatz* has the following local analog. Consider the vector space of k-jets of germs of

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analytic functions at a point u of the space of variables $\mathbb{C}^n$. Two k-jets are said to be *equivalent* if their difference coincides at the point u with the k-jet of some polynomial belonging to the ideal $\mathcal{I}$ of the algebraic variety. The dimension $HS_u(k)$ of the arising quotient space is the value at k of the Hilbert–Samuel function of the variety at the point u. The local version of the *Nullstellensatz* asserts that the function $HS_u(k)$ is a polynomial for sufficiently large positive integers. The degree of this polynomial is equal to the dimension r of the germ of M at the point u, and its leading coefficient multiplied by $r!$ is the multiplicity of the point u of the variety M (i.e., the multiplicity of the intersection of M with a general affine plane of complementary dimension at the point u). In this paper, we determine the relation between a system of differential equations and the Hilbert–Samuel polynomial of its symbol. The solutions of the form $P(x)e^{(u,x)}$, where $P(x)$ is a polynomial of degree at most k, constitute a vector space. We prove that the dimension of this space is equal to $HS_u(k)$ (Proposition 4).

In the case where the solution space of the system is finite-dimensional, as well as in the classical case of a linear ordinary differential equation with constant coefficients, all the solutions are quasipolynomial.

All arguments used in this paper are very simple and refer rather to algebraic geometry than to the theory of partial differential equations.

2. FORMAL SOLUTIONS OF THE SYSTEM

In this section, we identify the formal solutions of a system of linear partial differential equations having constant coefficients with linear functionals on the ring of differential operators and describe the spaces of formal and polynomial solutions of such a system.

Let us introduce the necessary notation. We consider systems of linear differential equations with constant coefficients for one unknown function:

$$\begin{cases} \displaystyle\sum_\alpha a^1_\alpha \partial_\alpha z(x) = 0, \\ \dots\dots\dots\dots\dots\dots \\ \displaystyle\sum_\alpha a^i_\alpha \partial_\alpha z(x) = 0, \\ \dots\dots\dots\dots\dots\dots \end{cases} \qquad a^i_\alpha \in \mathbb{C}. \tag{S}$$

Here and in what follows, $x = (x_1, \dots, x_n)$ belongs to the space $\mathbb{C}^n$ of independent variables, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$ is a multi-index, and

$$\partial_\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}, \qquad \text{where} \quad |\alpha| = \sum_{i=1}^n \alpha_i.$$

The number of equations in system (S) may be infinite.

First, note that, for systems with constant coefficients, the spaces of formal and analytic solutions in a neighborhood of a point u in the space of variables do not depend on this point. Indeed, if a series $f(x) \in \mathbb{C}[[x]]$ is a solution of the system, then, for any $u \in \mathbb{C}^n$, the series $f(x-u) \in \mathbb{C}[[x-u]]$ is a solution as well. We denote the vector space of formal solutions of system (S) by $F(S)$.

Let Dif_n denote the ring of differential operators with complex constant coefficients in the variables $x_1, \dots, x_n$; each operator $d \in \mathrm{Dif}_n$ has the form

$$d = \sum_{\alpha \in \operatorname{supp} d} d_\alpha \partial_\alpha, \tag{1}$$

where $\operatorname{supp} d \subset \mathbb{Z}^n_{\geq 0}$ is a finite set and the coefficients d_α are nonzero complex numbers. The set $\operatorname{supp} d$ is called the *support* of the operator d. Formula (1) identifies the space Dif_n with

$\mathbb{C}[\partial/\partial x_1, \dots, \partial/\partial x_n]$. Moreover, since we deal with operators with constant coefficients, Eq. (1) establishes the ring isomorphism

$$\mathrm{Dif}_n \simeq \mathbb{C}\left[\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right]$$

(i.e., the composition of operators coincides with the product of the corresponding polynomials). Let $\widehat{\mathrm{Dif}}_n$ be the ring of formal differential operators in the variables $x_1, \dots, x_n$. A formal differential operator is a series of the form

$$d = \sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} d_\alpha \partial_\alpha,$$

where $d_\alpha \in \mathbb{C}$. The ring structure is determined by the composition of operators. For the ring of formal operators with constant coefficients, we have the natural ring isomorphism

$$\widehat{\mathrm{Dif}}_n \simeq \mathbb{C}\left[\left[\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right]\right].$$

We consider Dif_n and $\widehat{\mathrm{Dif}}_n$ as infinite-dimensional complex topological vector spaces with the topology of pointwise convergence. Let V be a topological vector space. By V^* we denote the space dual to V.

Lemma 1. *The following natural isomorphisms hold*:

$$(\mathrm{Dif}_n)^* \simeq \mathbb{C}[[x]], \tag{2}$$

$$(\widehat{\mathrm{Dif}}_n)^* \simeq \mathbb{C}[x]. \tag{3}$$

Moreover, the natural embeddings $\mathrm{Dif}_n \to \widehat{\mathrm{Dif}}_n$ *and* $\mathbb{C}[x] \to \mathbb{C}[[x]]$ *are conjugate.*

Proof. Obviously, the isomorphism (2) is established by the map

$$f\colon \mathbb{C}[[x]] \to (\mathrm{Dif}_n)^*, \qquad f(d) = d(f)|_0, \tag{4}$$

where $f, d(f) \in \mathbb{C}[[x]]$, $d \in \mathrm{Dif}_n$, and $d(f)|_0$ denotes the free term of the series $d(f)$. Let us verify that the isomorphism (3) is established by the map

$$\mathbb{C}[x] \to (\widehat{\mathrm{Dif}}_n)^*, \qquad p(d) = d(p)(0), \tag{5}$$

where $p, d(p) \in \mathbb{C}[x]$ and $d \in \widehat{\mathrm{Dif}}_n$. The injectivity of (5) is obvious. Let us prove its surjectivity. Let $l \in (\widehat{\mathrm{Dif}}_n)^*$ be a continuous functional. We claim that $l(\partial_\alpha)$ is nonzero only for a finite number of $\alpha \in \mathbb{Z}^n_{\geq 0}$ (this implies the surjectivity of (5)). Indeed, suppose that, on the contrary, $l(\alpha_i) = l_i \neq 0$, where $i = 1, 2, \dots$ and $\alpha_i \neq \alpha_j$ for $i \neq j$. Obviously, the sequence $\{\frac{i}{l_i}\partial_{\alpha_i}\}$ converges to 0 in $\widehat{\mathrm{Dif}}_n$, while $\{l(\frac{i}{l_i}\partial_{\alpha_i}) = i\}$ diverges. This contradiction shows that the map (5) is surjective. The last assertions of the lemma follow from the explicit formulas (4) and (5). □

Let $\mathcal{I}$ denote the ideal of the ring Dif_n generated by the operators on the left-hand sides of the equations of system (S). We establish the correspondence between the formal solutions of the system and the linear functionals on the ring of differential operators.

Proposition 1. *A formal series* $f \in \mathbb{C}[[x]]$ *determines a linear functional on* Dif_n, *which vanishes identically on the ideal* $\mathcal{I}$ *if and only if the formal series* $d(f)$ *is identically zero for any operator* $d \in \mathcal{I}$.

Proof. Indeed, for any operator $d \in \mathrm{Dif}_n$, the equality $d(f) = 0$ is equivalent to

$$\partial_\alpha\big(d(f)\big)\big|_0 = (\partial_\alpha \circ d)(f)|_0 = 0$$

for all $\alpha \in \mathbb{Z}^n_{\geq 0}$. $\square$

Proposition 1 has the following corollary.

Corollary 1. *The space $F(S)$ of formal solutions of system (S) is naturally isomorphic to the vector space of linear functionals on the ring Dif_n vanishing on the ideal $\mathcal{I}$.*

Suppose that $\mathrm{Dif}_n = KI \oplus \mathcal{I}$, where KI is a vector subspace transversal to $\mathcal{I}$. Yet another reformulation of Proposition 1 is the following corollary.

Corollary 2. *The following isomorphisms of vector spaces hold*:

$$F(S) \simeq (KI)^* \simeq (\mathrm{Dif}_n/\mathcal{I})^*. \tag{6}$$

Now, let us describe the space $A_0(S)$ of polynomial solutions of the system.

Lemma 2. *The following natural isomorphism holds*:

$$A_0(S) \simeq (\widehat{\mathrm{Dif}}_n/\mathcal{I}\cdot\widehat{\mathrm{Dif}}_n)^*. \tag{7}$$

Here $\mathcal{I}\cdot\widehat{\mathrm{Dif}}_n$ denotes the minimal ideal in $\widehat{\mathrm{Dif}}_n$ containing $\mathcal{I}$ (the ring Dif_n is treated as a subring of $\widehat{\mathrm{Dif}}_n$).

Proof. By virtue of Lemma 1, we can identify the space $(\widehat{\mathrm{Dif}}_n/\mathcal{I}\cdot\widehat{\mathrm{Dif}}_n)^*$ with a vector subspace in the polynomial ring $\mathbb{C}[x]$. Comparing (4) and (5), we see that

$$A_0(S) = (\widehat{\mathrm{Dif}}_n/\mathcal{I}\cdot\widehat{\mathrm{Dif}}_n)^*$$

under this identification. $\square$

In Lemma 1 and in what follows, it is important that all vector spaces naturally identified with spaces of formal series are assumed to be endowed with the topology of pointwise convergence, and their quotient spaces, with the quotient topology.

3. SYMBOL OF A SYSTEM AS AN ALGEBRAIC VARIETY

In this section, we recall the definition of the symbol of the system of equations (S) and some algebraic notions used in what follows.

Let $(\mathbb{C}^n)^* = \{\xi = (\xi_1, \dots, \xi_n) \mid (\xi_i, x_j) = \delta_i^j\}$ be the space dual to the space of independent variables.

Consider the ring isomorphism

$$\mathrm{pr}\colon \widehat{\mathrm{Dif}}_n \to \mathbb{C}[[\xi_1, \dots, \xi_n]], \qquad \mathrm{pr}(\partial_\alpha) = \xi^\alpha. \tag{8}$$

The *symbol* (or *complete symbol*) of system (S) is defined as the affine algebraic set (variety) M in $(\mathbb{C}^n)^*$ corresponding to the ideal $\mathrm{pr}(\mathcal{I})$ of the ring $\mathbb{C}[\xi_1, \dots, \xi_n]$. By an *affine algebraic variety* M we mean a pair $\langle \widetilde{M}, \mathrm{pr}(\mathcal{I})\rangle$, where $\widetilde{M}$ is the set of common zeros of the polynomials from the ideal $\mathrm{pr}(\mathcal{I})$.

This definition is not the usual one, but it is most convenient for the purposes of this paper. Generally, the algebraic variety M is singular and reducible and has embedded components. If the ideal $\mathrm{pr}(\mathcal{I})$ is prime, we obtain the classical definition of an affine algebraic variety (see, e.g., [1]).

All algebraic notions and assertions which we use have natural generalizations to the case under consideration (see [1, 2]).

Recall some necessary notions from algebraic geometry. The *affine coordinate ring* (or the *ring of regular functions*) of an algebraic variety M is defined as $R_M := \mathbb{C}[\xi]/\operatorname{pr}(\mathcal{I})$ (accordingly, its elements are called *regular functions*).

The *local ring* of a point δ in a variety M is, by definition, the localization of the affine coordinate ring R_M with respect to the multiplicative system of regular functions not vanishing at δ (see, e.g., [1]). We denote this ring by $\mathcal{O}_{\delta,M}$. We use $\widehat{\mathcal{O}}_{\delta,M}$ to denote the formal completion of $\mathcal{O}_{\delta,M}$ with respect to the maximal ideal (see, e.g., [1]). The following expression can be regarded as a definition of the ring $\widehat{\mathcal{O}}_{\delta,M}$:

$$\widehat{\mathcal{O}}_{\delta,M} = \mathbb{C}[[\xi-\delta]]/\operatorname{pr}(\mathcal{I})\cdot\mathbb{C}[[\xi-\delta]].$$

In (15), $\operatorname{pr}(\mathcal{I})\cdot\mathbb{C}[[\xi-\delta]]$ denotes the minimal ideal of the ring $\mathbb{C}[[\xi-\delta]]$ of formal series centered at ξ that contains all polynomials from $\operatorname{pr}(\mathcal{I})$ (we assume the ring of polynomials $\mathbb{C}[\xi]$ to be naturally embedded in the ring of formal series $\mathbb{C}[[\xi-\delta]]$).

For each point $\delta \in M$, a natural ring homomorphism

$$R_M \to \widehat{\mathcal{O}}_{\delta,M} \tag{9}$$

is defined. Indeed, each polynomial is a function defined on the entire affine space; therefore, we can expand it in a series in a neighborhood of δ and consider as an element of the ring $\widehat{\mathcal{O}}_{\delta,M}$.

4. SYMBOL OF A SYSTEM AND FORMAL SOLUTIONS

In this section, we interpret the formal solutions of system (S) as linear functionals on the affine coordinate ring of the algebraic variety M. We also study spaces of jets of formal solutions.

Obviously, the isomorphism (6) can be interpreted in terms of the affine coordinate ring of the variety M as follows.

Lemma 3. *The following isomorphism holds*:

$$F(S) \simeq R_M^*. \tag{10}$$

For a nonnegative integer i, we set

$$F_i(S) = F(S)/\big(f \sim g \overset{\text{def}}{\Longleftrightarrow} f-g = o(x^i)\big)\,; \tag{11}$$

this is the space of i-jets of formal solutions. The expression $o(x^i)$ is understood as the series of monomials x^α whose exponents satisfy the inequality $|\alpha| > i$.

We also set

$$R_i = \big\{[p] \in R_M \mid p \in \mathbb{C}[x],\ \deg p < i+1\big\}. \tag{12}$$

The spaces $F_i(S)$ and R_i are related as follows.

Lemma 4. *The vector spaces $F_i(S)$ and R_i^* are isomorphic for each i*:

$$F_i(S) \cong R_i^*. \tag{13}$$

Proof. Indeed, the choice of an i-jet of a formal solution f of the system is equivalent to the specification of the values of the corresponding functional for the operators ∂_α with $|\alpha| \le i$ ($\alpha \in \mathbb{Z}^n_{\ge 0}$), i.e., on the subspace $R_i \subset R_M$ (see (4)), which implies the required assertion. $\square$

The function $FH(i) = \dim F_i(S)$ of the positive integer argument i is called the *Hilbert function of system* (S).

Recall that the *Hilbert function of the algebraic variety* M is defined by $H(i) = \dim R_i$.

By virtue of (13), we have $\dim F_i(S) = \dim R_i$. Thus, the following proposition is valid.

Proposition 2. *The Hilbert function of system* (S) *coincides with the Hilbert function of the symbol* M *of this system. In particular, the Hilbert function* FH *of the system is a polynomial on the set of sufficiently large positive integers* i. *Moreover, the degree* r *of this polynomial is equal to the dimension of the symbol* M *of the system, and its leading coefficient is* $\deg \overline{M}/r!$.

By $\deg \overline{M}$ we denote the degree of the closure of M in $\mathbb{CP}^n$ under the standard embedding.

5. SYMBOL OF A SYSTEM AND ANALYTIC SOLUTIONS

In this section, we study the relationship between the spaces of formal and analytic solutions of the system under consideration. With the symbol M of the system we associate some special class of its analytic solutions and prove a theorem about the approximation of formal solutions by analytic solutions from this special class.

For a point δ of the space $(\mathbb{C}^n)^*$, let $A_\delta(S)$ denote the vector space of solutions of (S) having the form

$$p(x)e^{(\delta,x)}, \tag{14}$$

where $p \in \mathbb{C}[x]$ is a polynomial.

The following proposition describes the spaces $A_\delta(S)$.

Proposition 3. *The following natural identifications hold*:

$$A_\delta(S) \simeq \big(\mathbb{C}[[\xi-\delta]]/\operatorname{pr}(\mathcal{I})\cdot\mathbb{C}[[\xi-\delta]]\big)^* = (\widehat{\mathcal{O}}_{\delta,M})^*. \tag{15}$$

Moreover, the natural map $A_\delta(S) \to F(S)$ *is conjugate to the natural map* $R_M \to \widehat{\mathcal{O}}_{\delta,M}$.

Proof. For $\delta = 0$, the proposition (identifications (15)) is an obvious reformulation of Lemma 2. For an arbitrary δ, (15) can be reduced to the case $\delta = 0$ by applying the formula

$$d\biggl(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}\biggr)P(x)e^{(\delta,x)} = e^{(\delta,x)}d\biggl(\frac{\partial}{\partial x_1}+\delta_1,\dots,\frac{\partial}{\partial x_n}+\delta_n\biggr)f(x). \tag{16}$$

Here f is a holomorphic function, $\delta = (\delta_1,\dots,\delta_n)$ is a point of $(\mathbb{C}^n)^*$, and $d(\partial/\partial x_1,\dots,\partial/\partial x_n)$ is a differential operator (d is a polynomial in $\partial/\partial x_1$).

Let us prove the second part of the proposition. We denote the natural map $R_M \to \widehat{\mathcal{O}}_{\delta,M}$ by f and the natural map $A_\delta(S) \to F(S)$ by g. For $\varphi \in A_\delta(S)$ and $d \in R_M$, we have

$$\big(g(\varphi)\big)(d) = \operatorname{pr}^{-1}(d)\big(g(\varphi)\big)\big|_0 = \big(\operatorname{pr}^{-1}(d)(\varphi)\big)(0) = \varphi\big(g(d)\big),$$

as required. $\square$

Corollary 3. *System* (S) *has nontrivial solutions of the form* (14) *if and only if the point* δ *belongs to the variety* M.

Proof. Indeed, if δ belongs to M, then all polynomials from the ideal $\mathrm{pr}(\mathcal{I})$ vanish at δ; therefore, the ideal $\mathrm{pr}(\mathcal{I})\cdot\mathbb{C}[[\xi-\delta]]$ does not contain 1, and the quotient space is different from 0. In other words, the function $e^{(\delta,x)}$ is necessarily a solution in this case.

If δ does not belongs to M, then the ideal $\mathrm{pr}(\mathcal{I})\cdot\mathbb{C}[[\xi-\delta]]$ contains 1 and, therefore, coincides with the entire space $\mathbb{C}[[\xi-\delta]]$. $\square$

For integer nonnegative i, we set, by analogy with (11),

$$A_{\delta,i}(S)=A_\delta(S)/\big(f\sim g \overset{\text{def}}{\iff} f-g=o(x^i)\big)\,; \tag{17}$$

this is the space of i-jets at zero of the solutions of the form $p(x)e^{(\delta,x)}$, where δ is a point of the space $(\mathbb{C}^n)^*$. Let

$$[\widehat{\mathcal{O}}_{\delta,M}]_i=\widehat{\mathcal{O}}_{\delta,M}/\mathcal{M}_\delta^{i+1}, \tag{18}$$

where $\mathcal{M}_\delta$ denotes the maximal ideal of the local ring $\widehat{\mathcal{O}}_{\delta,M}$. The following lemma establishes a relationship between the spaces $A_{\delta,i}(S)$ and $[\widehat{\mathcal{O}}_{\delta,M}]_i$.

Lemma 5. *For each i, the following isomorphism holds*:

$$A_{\delta,i}(S)\cong[\widehat{\mathcal{O}}_{\delta,M}]_i^*. \tag{19}$$

Proof. The choice of an i-jet of a quasi-exponential solution at a point $\delta\in M$ is equivalent to the specification of the linear functional (on the space of formal differential operators centered at δ) corresponding to this solution for the operators $(\partial/\partial x-\xi)^\alpha$ with $|\alpha|\le i$ $(\alpha\in\mathbb{Z}^n_{\ge 0})$. $\square$

For each point $\delta\in(\mathbb{C}^n)^*$, we call the function $AHS_\delta(i)=\dim A_{\delta,i}(S)$ of the positive integer argument i the *Hilbert–Samuel function* of system (S) at the point δ.

Recall that, for each point δ of the variety M, the Hilbert–Samuel function of M is defined by $HS_\delta(i)=\dim[\widehat{\mathcal{O}}_{\delta,M}]_i$. It is natural to set the Hilbert–Samuel function to be identically zero for δ not belonging to M. Thus, we assume that the Hilbert–Samuel function HS is defined at all points of the space $(\mathbb{C}^n)^*$.

The isomorphism (19) implies $\dim A_{\delta,i}(S)=[\widehat{\mathcal{O}}_{\delta,M}]_i$; thus, we have obtained the following result.

Proposition 4. *For each point δ of the space $(\mathbb{C}^n)^*$, the Hilbert–Samuel function of system (S) coincides with the Hilbert–Samuel function of the symbol M of this system. In particular, for each $\delta\in(\mathbb{C}^n)^*$, the Hilbert–Samuel function $AHS_\delta(i)$ is a polynomial on the set of sufficiently large positive integers i. Moreover, the degree r of this polynomial is equal to the dimension of the symbol M of the system, and its leading coefficient is* $\mathrm{mult}_M\,\delta/r!$.

If a point δ does not belong to the variety M, then it is natural to assume its multiplicity to be zero. The smooth points of the variety have multiplicity 1.

Corollary 4. *For a smooth point δ of the symbol M of the system, the Hilbert–Samuel function of system (S) is*

$$AHS_\delta(i)=C^i_{n+i-1}. \tag{20}$$

Here and in what follows, the symbols C^k_n denote binomial coefficients.

Proof. For a smooth point δ of the symbol M, the ring $\widehat{\mathcal{O}}_{\delta,M}$ is isomorphic to the ring of power series in r variables (recall that r is the dimension of the algebraic variety M). Therefore, the value $HS_\delta(i)$ of the Hilbert–Samuel function is equal to the dimension of the space of polynomials of degree at most i in r variables. Thus, $HS_\delta(i)=C^i_{n+i-1}$, which implies the required assertion. $\square$

Now, let us prove a theorem about the approximation of formal solutions of system (S) by solutions of the form (14).

Let $\mathrm{pr}(\mathcal{I})=\mathcal{I}_1\cap\dots\cap\mathcal{I}_t$ be a primary decomposition of the ideal $\mathrm{pr}(\mathcal{I})$ (see [2]). Take t different points $\xi_1,\dots,\xi_t\in M$, where ξ_k belongs to the (possibly embedded) component of M corresponding to the primary ideal $\mathcal{I}_k$.

Theorem 1. *For any nonnegative integer m and formal solution*

$$f = \sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} f_\alpha x^\alpha,$$

there exist quasipolynomial solutions $P_1(x)e^{(\xi_1, x)}, \dots, P_t(x)e^{(\xi_t, x)}$ of the system such that

$$j_0^m f = \sum_{|\alpha| \leq m} f_\alpha x^\alpha = \sum_{i=1}^{t} j_0^m P_i(x) e^{(\xi_i, x)}. \tag{21}$$

Let $\mathcal{O}_i(S)$ be the space of i-jets at zero of the analytic solutions to the system. Theorem 1 has the following corollary.

Theorem 2. *For each i, the dimensions of the spaces $\dim \mathcal{O}_i(S)$ and $\dim F_i(S)$ coincide. Thus, the dependence of the dimension of $\mathcal{O}_i(S)$ on i is polynomial for sufficiently large i.*

Proof of Theorem 1. Take any point $\xi \in M$. We identify the spaces $F(S)$ and $A_\xi(S)$ with the dual spaces to R_M and $\widehat{\mathcal{O}}_{\xi, M}$, respectively. For each pair p, q of nonnegative integers such that $q \geq p$, the linear map

$$f_{p,q} \colon \bigoplus_j A_{\xi_j, q}(S) \to F_p(S) \tag{22}$$

induced by the natural map $\bigoplus_j A_{\xi_j}(S) \to F(S)$ is well defined (each quasi-exponential solution is expanded in a series in a neighborhood of 0, and the series are added together). According to the last assertion of Proposition 3, the adjoint map

$$f^*_{p,q} \colon R_p \to \bigoplus_j [\widehat{\mathcal{O}}_{\xi_i, M}]_q \tag{23}$$

is induced by the natural map $R_M \to \bigoplus_j \widehat{\mathcal{O}}_{\xi_i, M}$ (see (9)). For any p and q such that $q \geq p$, $f_{p,q}$ and $f^*_{p,q}$ are linear maps of finite-dimensional vector spaces; hence, if $f^*_{p,q}$ has trivial kernel, then the map $f_{p,q}$ is surjective. Thus, Theorem 1 is equivalent to the following assertion.

Lemma 6. *For each nonnegative integer m, there exists a nonnegative integer $N(m) \geq m$ such that the map $f^*_{m, N(m)}$ has trivial kernel.*

We have reduced Theorem 1 to a purely algebraic assertion about the structure of the algebraic variety M. Before proving the lemma for an arbitrary algebraic variety M, consider the case where the symbol M of the system is a connected smooth variety. In this case, $t = 1$ and ξ_1 is an arbitrary point of M. Take a nonnegative integer m and let p be an element of R_m. We can assume that p is simply a polynomial of degree at most m not vanishing identically on the variety M. Thus, the multiplicity $N(p) < \infty$ of the zero of p at the point ξ_1 is well defined. This multiplicity $N(p)$ is the maximal nonnegative integer such that the polynomial p belongs to the power $\widehat{\mathcal{M}}^{N(p)}_{\xi_1, M}$ of the maximal ideal of the local ring $\widehat{\mathcal{O}}_{\xi_1, M} \cong \mathbb{C}[[y_1, \dots, y_r]]$, where $y_1, \dots, y_r$ are local coordinates on the variety M in a neighborhood of ξ_1. Since $N(p) < \infty$ for each p, it follows that the natural linear map $R_M \to \widehat{\mathcal{O}}_{\xi_1, M}$ has trivial kernel. Obviously, the elements of the ring R_m that vanish at ξ_1 and have multiplicities not smaller that some number $i \in \mathbb{Z}_{\geq 0}$ form a vector subspace in R_m. We denote it by V_i. We have obtained an infinite decreasing filtration

$$R_m = V_0 \supseteq V_1 \supseteq V_2 \supseteq \cdots.$$

The space R_m is finite-dimensional; therefore, this filtration stabilizes starting with some number $N_0(m)$. We set $N(m) = \max(N_0(m), m)$. The natural map

$$R_m \to [\widehat{\mathcal{O}}_{\xi_1, M}]_{N(m)} \tag{24}$$

has trivial kernel, as required. The proof of the lemma in the general case is similar, but it requires applying some results from commutative algebra, which can be found in, e.g., [2].

Proof of Lemma 6. Let us prove that the natural map

$$R_M \to \bigoplus_1^t \widehat{\mathcal{O}}_{\xi_i,M} \tag{25}$$

has trivial kernel. Take $p \in \mathbb{C}[\xi] \setminus \mathrm{pr}(\mathcal{I})$. Since p does not belong to $\mathrm{pr}(\mathcal{I})$, there exists a k such that $1 \le k \le t$ and p does not belong to the primary ideal $\mathcal{I}_k$. Without loss of generality, we can assume that $k = 1$. Let $\mathcal{O}_{\xi_1,\mathbb{C}^n}$ be the ring of rational functions whose denominators do not vanish at the point ξ_1 (in other words, $\mathcal{O}_{\xi_1,\mathbb{C}^n}$ is the localization of the polynomial ring $\mathbb{C}[\xi]$ with respect to the complement of the prime ideal $\mathcal{M}_{\xi_1,\mathbb{C}^n}$ consisting of all polynomials vanishing at ξ_1). The polynomial ring $\mathbb{C}[\xi]$ is naturally embedded in its localization $\mathcal{O}_{\xi_1,\mathbb{C}^n}$. The primary ideal $\mathcal{I}_1$ is contained in $\mathcal{M}_{\xi_1,\mathbb{C}^n}$. Therefore, (see, e.g., [2]),

$$\mathcal{I}_1 \cdot \mathcal{O}_{\xi_1,\mathbb{C}^n} \cap \mathbb{C}[\xi] = \mathcal{I}_1\,;$$

on the other hand,

$$\mathrm{pr}(\mathcal{I}) \cdot \mathcal{O}_{\xi_1,\mathbb{C}^n} = (\mathcal{I}_1 \cdot \mathcal{O}_{\xi_1,\mathbb{C}^n}) \cap \dots \cap (\mathcal{I}_t \cdot \mathcal{O}_{\xi_1,\mathbb{C}^n}).$$

The polynomial p as an element of the ring $\mathcal{O}_{\xi_1,\mathbb{C}^n}$ does not belong to the ideal $\mathcal{I}_1 \cdot \mathcal{O}_{\xi_1,\mathbb{C}^n}$; hence p does not belong to the ideal $\mathrm{pr}(\mathcal{I}) \cdot \mathcal{O}_{\xi_1,\mathbb{C}^n}$.

Thus, each polynomial determining a nonzero element of the ring R_M determines a nonzero element of the local ring

$$\mathcal{O}_{\xi_k,M} = \mathcal{O}_{\xi_k,\mathbb{C}^n}/(\mathrm{pr}(\mathcal{I}) \cdot \mathcal{O}_{\xi_k,\mathbb{C}^n})$$

for some k. By a theorem of Krull (see, e.g., [2]), the natural homomorphism

$$\mathcal{O}_{\xi,M} \to \widehat{\mathcal{O}}_{\xi,M}\,,$$

where $\xi \in M$ is an arbitrary point, is injective. Therefore, the natural map takes each nonzero element of the ring R_M to a nonzero element of the ring $\widehat{\mathcal{O}}_{\xi_k,M}$ for some k such that $1 \le k \le t$, which implies (25).

A word for word repetition of the argument used in the smooth case (which reduces to constructing a decreasing filtration by vector subspaces of a finite-dimensional vector space) proves that, for each nonnegative integer m, there exists a nonnegative integer $N(m) \ge m$ such that the natural map

$$R_m \to \bigoplus_j [\widehat{\mathcal{O}}_{\xi_i,M}]_{N(m)}$$

has trivial kernel, as required. $\square$

This completes the proof of the theorem. $\square$

6. AN EXAMPLE: HARMONIC FUNCTIONS

Suppose that system (S) consists of one equation

$$\Delta z = 0, \tag{26}$$

where

$$\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$$

is the Laplace operator. In this case, the ideal $\mathrm{pr}(\mathcal{I})$ is prime and

$$M = \{\xi_1^2 + \dots + \xi_n^2 = 0\} \tag{27}$$

is a classical algebraic variety. All the points of M, except the origin, are smooth. Therefore, by Corollary 4, we have

$$HS_\delta(i) = C_{n+i-1}^i, \tag{28}$$

where $n, i > 1$ and $\delta(\neq 0) \in M$.

An easy calculation gives

$$H(i) = HS_0(i) = C_{n+i-1}^i + C_{n+i-1}^{i-1} = \frac{2}{(n-1)!}\, i^{(n-1)} + \{\text{lower-order terms}\} \tag{29}$$

for $i > 2$.

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